Worcester County Mathematics League

Varsity Meet 2 - December 7, 2022

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 2 - December 7, 2022
Answer Key

Round 1 - Fractions, Decimals, and Percents

1. 7.8
2. $0.\overline{000999}$ or $0.000\overline{999000}$, or equivalent
3. 13

Round 2 - Algebra I

1. -10
2. 160
3. 17

Round 3 - Parallel Lines and Polygons

1. 50
2. 174
3. 15

Round 4 - Sequences and Series

1. $\{2, 9\}$ or 2, 9 (need both, either order)
2. 90
3. 31

Round 5 - Matrices and Systems of Equations

1. $\begin{bmatrix} 5 & 4 & 26 \\ 2 & 3 & 16 \end{bmatrix}$
2. -12
3. $(6, -\dfrac{59}{3}, \dfrac{52}{3})$ or $(6, -19\frac{2}{3}, 17\frac{1}{3})$ or $(6, -19.\overline{6}, 17.\overline{3})$
all three numbers, in exact order

Team Round

1. 273
2. 150
3. 72
4. 101
5. $\begin{bmatrix} -1 & 0 \\ -24 & 5 \end{bmatrix}$
6. 14
7. $(-32, 32, 2)$ all three, in exact order
8. 295
9. 724

Worcester County Mathematics League
Varsity Meet 2 - December 7, 2022
Round 1 - Fractions, Decimals, and Percents

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. A sweater is 65% wool by weight. If the sweater weighs 12 ounces, what is the weight of wool in the sweater, in ounces (oz.)? Express your answer as a decimal.

2. Find the decimal form for the fraction $\frac{1}{1001}$. Express your answer using repeating notation. For instance, the decimal form for $\frac{1}{3}$ is $0.\overline{3}$.

3. A box contained 31 chocolates. Susan and Jacqueline ate all of the chocolates in two days. The first day Susan ate $\frac{3}{4}$ of the number of chocolates that Jacqueline ate that day. The second day Susan ate $\frac{2}{3}$ of the number that Jacqueline ate that second day. How many of the 31 chocolates did Susan eat in the two days?

ANSWERS

(1 pt) 1. ______________________ oz.

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Hopkinton, QSC, Bay Path RVTHS

Worcester County Mathematics League
Varsity Meet 2 - December 7, 2022
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Given the system of equations shown below, find $x + y$:

$$\begin{cases} 2x + y = 10 \\ x + 2y = -40 \end{cases}$$

2. The highway between city A and city B consists of two segments, one 96km longer than the other. A car averages 60 km/h over the shorter segment, 120 km/h over the longer segment, and 100 km/h over the entire trip. How far apart, in km, are city A and city B?

3. At a WoCoMaL meet there are 10 times as many boys as coaches, and the number of boys is 50 less than twice the number of girls. If there are no more than 300 people (boys, girls, and coaches) attending the meet, what is the greatest possible number of coaches at the meet?

ANSWERS

(1 pt) 1. $x + y =$ ________________

(2 pts) 2. ________________________ km

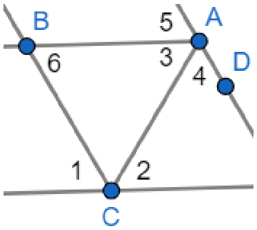
(3 pts) 3. ________________________

Doherty, Tantasqua, Notre Dame

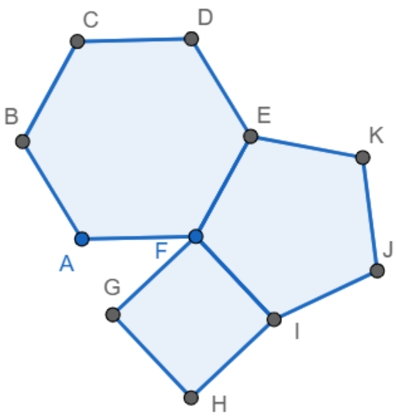
Worcester County Mathematics League
Varsity Meet 2 - December 7, 2022
Round 3 - Parallel Lines and Polygons

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

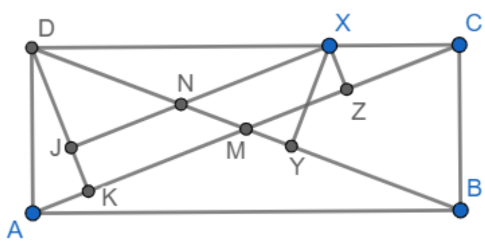
1. Given the figure below, with $AB = AC$, $AD||BC$, and $m\angle 6 = 50°$, find $m\angle 4$, expressed in degrees.



2. In the figure below, $ABCDEF$ is a regular hexagon, $EFIJK$ is a regular pentagon, and $FGHI$ is a square. Find $m\angle AFG + m\angle DEK$. Express your answer in degrees.



3. Given rectangle $ABCD$ in the figure below, where $\overline{XY} \perp \overline{BD}$, $\overline{XZ} \perp \overline{AC}$, $\overline{DK} \perp \overline{AC}$, $\overline{XJ} \perp \overline{DK}$, $DK = 60$, $XY = 45$ and $BC = 65$. Find XZ.



ANSWERS

(1 pt) 1. ________________ °

(2 pts) 2. ________________ °

(3 pts) 3. ________________

Uxbridge, St. John's, Southbridge

Worcester County Mathematics League
Varsity Meet 2 - December 7, 2022
Round 4 - Sequences and Series

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find all values of n such that the sum of the first n terms of sequence $10, 8, 6, 4, ...$ equals 18.

2. A ball is dropped from 10 feet above the ground. Each time it bounces, it rebounds to $\frac{4}{5}$ of the height from which it fell. How far, in feet, will the ball travel before coming to rest?

3. How many terms of the arithmetic sequence $1, \frac{17}{15}, \frac{19}{15}, ...$ must be summed to equal the sum of the first five terms of the geometric sequence $3, 6, 12, ...$?

ANSWERS

(1 pt) 1. $n \in \{$____________________$\}$

(2 pts) 2. ____________________ feet

(3 pts) 3. ____________________

Marlboro, Notre Dame, Worcester Academy

Round 5 - Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, find AB.

2. Find the determinant of $A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & -1 \\ 1 & 1 & 2 \end{bmatrix}$.

3. Solve the following system of linear equations. Write your solution as the ordered triple (x, y, z).

$$\begin{cases} -2x + y + 2z = 3 \\ -4x - 2y - z = -2 \\ 6x + 8y + 7z = 0 \end{cases}$$

ANSWERS

(1 pt) $AB = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

(2 pts) 2. det $A =$ ________________

(3 pts) 3. $(x, y, z) =$ (________________)

Holy Name, Southbridge, North

Worcester County Mathematics League
Varsity Meet 2 - December 7, 2022
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Joshua sold a bicycle for \$231, making a 10% profit on what he paid for the bike. At what price, in \$, would he have needed to sell it to make a 30% profit instead?

2. Tim paints fences at a rate of once fence every 5 hours. Jim paints fences at a rate of one fence every 4 hours. When Tim and Jim work together, Jim works at his usual rate, but Tim wastes time and paints fewer fences than he would at his usual rate. If Tim and Jim painted 4 fences in 10 hours working together, how much time did Tim waste? Express your answer in minutes.

3. The measure in degrees of an interior angle of a regular polygon is 90° less than 8 times the measure of an exterior angle. Each of the polygon's legs is 6 cm long. Find the perimeter of the polygon, in cm.

4. Find the product defined below and express your answer in simplest form.

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdot \ldots \cdot\left(1+\frac{1}{100}\right)$$

5. Let $A = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 4 & 5 \end{bmatrix}$. Find the matrix product $A^{-1}BA$, expressed as a single matrix.

6. A rabbit family consists of males and females. Each male in the family has one fewer female relatives than he has male relatives. Each female has two fewer male relatives than twice the number of her female relatives. What is the total number of rabbits (males and females) in the family?

7. A miniature stop sign in the shape of a regular octagon can be cut from a 4in X 4in square sheet of metal. The maximum area for the stop sign can be expressed as $a + b\sqrt{c}$ in^2. Find the ordered triple (a, b, c).

8. A contest winner was offered the choice of two prizes. The first prize consisted of 12 daily cash payments: \$160 on the first day, \$190 on the second day, \$220 on day 3, with the daily cash payment increasing by \$30 each day up to day 12. The other prize consisted of 12 daily cash payments: \$1 on day 1, \$2 on day 2, \$4 on day 3, doubling each day up to day 12. What is the absolute value of the difference (in \$) between the two prizes?

9. At the Pittsburgh zoo, children ride a train for \$0.50 and adults ride the train for \$2. On a given day, 1088 passengers paid a total of \$1090. How many of those passengers were children?

Shrewsbury, Hopkinton, Algonquin, Hudson, Worc. Acad., Algonquin, Wachusett, Shrewsbury, Tahanto

Worcester County Mathematics League
Varsity Meet 2 - December 7, 2022
Team Round Answer Sheet

ANSWERS

1. $ ________________________

2. ________________________ minutes

3. ________________________ cm

4. ________________________

5. $A^{-1}BA = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

6. ________________________

7. $(a, b, c) = ($________________$)$

8. $ ________________________

9. ________________________

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8. 295
9. 724

Round 1 - Fractions, Decimals, and Percents

1. A sweater is 65% wool by weight. If the sweater weighs 12 ounces, what is the weight of wool in the sweater, in ounces? Express your answer as a decimal.

Solution: Percent means "per hundred", so $65\% = \frac{65}{100} = 0.65$. Then the wool in the sweater weighs 0.65 times the sweater weight, and $12 \cdot 0.65 = 10 \cdot .65 + 2 \cdot .65 = 6.5 + 1.3 = \boxed{7.8}$ oz.

2. Find the decimal form for the fraction $\frac{1}{1001}$. Express your answer using repeating notation. For instance, the decimal form for $\frac{1}{3}$ is $0.\overline{3}$.

Solution: A repeating decimal is found by dividing the numerator by the denominator and noting when the remainder either matches the numerator or matches the remainder from a previous step. The same sequence of digits will repeat with subsequent steps. Applying long division to $\frac{1|}{1001}$:

$$\begin{array}{r} 0.000999 \\ 1001\overline{)1.000000} \\ \underline{9009} \\ 9910 \\ \underline{9009} \\ 9010 \\ \underline{9009} \\ 1 \end{array}$$

Note that the remainders at the three steps are 991, 901, and 1, so that the pattern "000999" will repeat. Therefore the decimal representation is $\boxed{0.\overline{000999}}$.

3. A box contained 31 chocolates. Susan and Jacqueline ate all of the chocolates in two days. The first day Susan ate $\frac{3}{4}$ of the number of chocolates that Jacqueline ate that day. The second day Susan ate $\frac{2}{3}$ of the number that Jacqueline ate that second day. How many of the 31 chocolates did Susan eat in the two days?

Solution: Let

- s_1 = the number of chocolates that Susan ate on day 1
- s_2 = the number of chocolates that Susan ate on day 2
- j_1 = the number of chocolates that Jacqueline ate on day 1
- j_2 = the number of chocolates that Jacqueline ate on day 2

and translate the given information into equations: $s_1 = \frac{3}{4}j_1$, $s_2 = \frac{2}{3}j_2$, and $s_1 + j_1 + s_2 + j_2 = 31$ because 31 chocolates were eaten in total. Substitute for s_1 and s_2 in the third equation using the first two equations:

$$\frac{3}{4}j_1 + j_1 + \frac{2}{3}j_2 + j_2 = 31$$
$$\frac{7}{4}j_1 + \frac{5}{3}j_2 = 31$$
$$21j_1 + 20j_2 = 31 \cdot 12 = 372$$

where the coefficients were converted the coefficients to integers in the last step after multiplying both sides of the equation by 12 = LCD(3,4).

Next, solve the Diophantine equation $21j_1 + 20j_2 = 372$ for integer j_1 and j_2 values. Note that $20j_2$ is a multiple of ten and its ones digit is 0. Therefore the ones digit of $21j_1$ must be 2, so that j_1 must be 2, 12, 22, or some other number ending in 2. Substituting $j_1 = 2$ into the Diophantine equation yields $42+20j_2 = 372$, which does not have an integer solution. Substituting $j_1 = 12$ into the Diophantine equation yields $252 + 20j_2 = 372$ so that $j_2 = \frac{372-252}{20} = \frac{120}{20} = 6$, and the solution is $j_1 = 12, j_2 = 6$. Finally, Susan ate $s_1 + s_2 = \frac{3}{4}j_1 + \frac{2}{3}j_2 = \frac{3}{4}12 + \frac{2}{3}6 = 9 + 4 = \boxed{13}$ chocolates.

Round 2 - Algebra I

1. Given the system of equations shown below, find $x + y$:

$$\begin{cases} 2x + y = 10 \\ x + 2y = -40 \end{cases}$$

Solution: The simplest solution begins by adding the two equations together, yielding

$$3x + 3y = -30.$$

The next step is to factor 3 from the left side of the equation, leaving

$$3(x + y) = -30.$$

Finally, divide both sides of the equation by 3, leaving: $x + y = \boxed{-10}$

Alternative solution: A straightforward, but longer, method solves for x and y individually using elimination and substitution. Subtract 2 times the second equation from the first equation, or equivalently add -2 times the second equation to the first equation:

$$\begin{array}{r} 2x + y = 10 \\ -2x - 4y = 80 \\ \hline -3y = 90 \end{array}$$

Divide both sides by -3 to find $y = -30$. Substitute this value into the second (original) equation, resulting in $x + 2(-30) = x - 60 = -40$. Solve for x and $x = 20$, so that $x + y = 20 - 30 = -10$.

2. The highway between city A and city B consists of two segments, one 96km longer than the other. A car averages 60 km/h over the shorter segment, 120 km/h over the longer segment, and 100 km/h over the entire trip. How far apart, in km, are city A and city B?

Solution: Let x equal the distance between cities A and B, and let y be the length of the shorter segment. Then the length of the longer segment is $y + 96$ and $x = y + y + 96 = 2y + 96$. Solve for y and

$$y = \frac{x - 96}{2}$$

. The rate equation, rate = distance/time, can also be written time = rate/distance. Using the second form of the rate equation, the car spent $\frac{y}{60}$ hours travelling on the shorter segment, $\frac{y+96}{120}$ hours on the longer segment, and $\frac{x}{100}$ hours total driving between cities A and B. The total time is equal to the sum of the times spent traversing the two segments so that:

$$\begin{aligned} \frac{x}{100} &= \frac{y}{60} + \frac{y+96}{120} \\ \frac{2y+96}{100} &= \frac{2y}{120} + \frac{y+96}{120} \\ \frac{y+48}{50} &= \frac{3y+96}{120} \\ &= \frac{y+32}{40} \end{aligned}$$

Next, multiply both sides of the equation by 200 and:

$$\begin{aligned} 4(y+48) &= 5(y+32) \\ 4y + 192 &= 5y + 160 \\ 192 - 160 &= 5y - 4y \\ 32 &= y \end{aligned}$$

Now substitute $\frac{x-96}{2}$ for y and solve for x.

$$\begin{aligned} y &= 32 \\ \frac{x-96}{2} &= 32 \\ x - 96 &= 64 \\ x = 64 + 96 = \boxed{160} \text{ km} \end{aligned}$$

Note that the method shown above (substituting for x, solving for y, and then finding x) is a little easier than substituting for y and solving for x directly.

3. At a WoCoMaL meet there are 10 times as many boys as coaches, and the number of boys is 50 less than twice the number of girls. If there are no more than 300 people (boys, girls, and coaches) attending the meet, what is the greatest possible number of coaches at the meet?

Solution: Let

- g equal the number of girls
- b equal the number of boys
- c equal the number of coaches

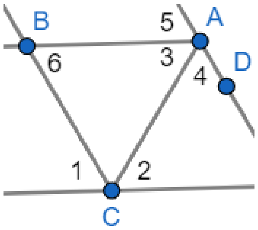
Then $b = 10c$, $b = 2g - 50$, and $g + b + c < 300$. Now b is already expressed in terms of c. Expressing g in terms of c and subsituting for b and g results in an inequality that can be used to find the maximum value for c. Specifically, $2g = b + 50 = 10c + 50$, and $g = 5c + 25$. Rewrite the inequality in terms of c:

$$\begin{aligned} 5c + 25 + 10c + c &< 300 \\ 16c &< 300 - 25 \\ c &< \frac{275}{16} = 17 + \frac{3}{16} \end{aligned}$$

and the maximum number of coaches is $\boxed{17}$.

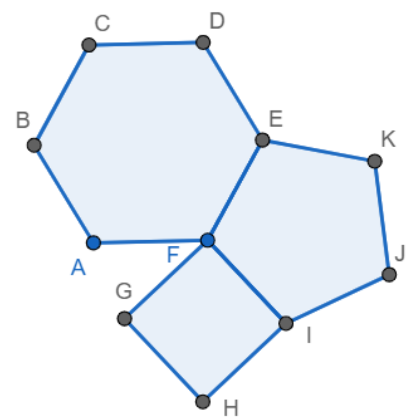
Round 3 - Parallel Lines and Polygons

1. Given the figure below, with $AB = AC$, $AD||BC$, and m$\angle 6 = 50°$, find m$\angle 4$, expressed in degrees.



Solution: First, $\angle 4 \cong \angle ACB$ because they are alternate interior angles and $\overline{AD}||\overline{BC}$. Next, $\overline{AB} \cong \overline{AC}$ (because $AB = AC$), and $\angle 6 \cong \angle ACB$ by the Isosceles Triangle Theorem. By the transitive property, $\angle 4 \cong \angle 6$ and so m$\angle 4 = \boxed{50}^{\circ}$.

2. In the figure at the right, $ABCDEF$ is a regular hexagon, $EFIJK$ is a regular pentagon, and $FGHI$ is a square. Find m$\angle AFG$ + m$\angle DEK$. Express your answer in degrees.

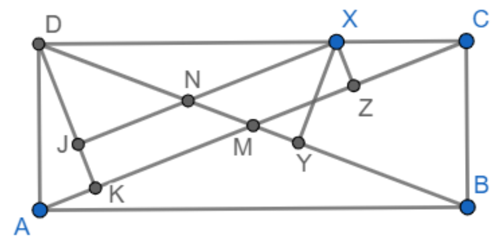


Solution: The measure of an interior angle of a regular polygon with n sides is $\frac{180°(n-2)}{n}$ Therefore the measure of an interior angle in a regular hexagon, a regular pentagon, and a square is equal to $\frac{180 \cdot 4}{6} = 120°$, $\frac{180 \cdot 3}{5} = 108°$, and $\frac{180 \cdot 2}{4} = 90°$, respectively.

Note that $\angle DEK$ shares vertex E with an an interior hexagon angle and an interior pentagon angle, and that the three angles together divide the entire plane. Therefore m$\angle DEK + 120 + 108 = 360$, and m$\angle DEK = 360 - 120 - 108 = 132°$.

Likewise, $\angle AFG$ and interior angles of the hexagon, pentagon, and square divide the plane, so that m$\angle AFG + 120 + 108 + 90 = 360$, and m$\angle AFG = 360 - 120 - 108 - 90 = 132 - 90 = 42°$. Finally, m$\angle AFG$ + m$\angle DEK = 132 + 42 = \boxed{174}^{\circ}$.

3. Given rectangle $ABCD$ in the figure at the right, where $\overline{XY} \perp \overline{BD}$, $\overline{XZ} \perp \overline{AC}$, $\overline{DK} \perp \overline{AC}$, $\overline{XJ} \perp \overline{DK}$, $DK = 60$, $XY = 45$ and $BC = 65$. Find XZ.



Solution: This diagram is filled with several similar and congruent right triangles, and it is not obvious how to find XZ from the given information.

Start by observing that quadrilateral $JKZX$ is a rectangle. (The angles at J, K, and Z are created by perpendicular lines as given and have measure $90°$. Since the measures of the angles in $JKZX$ must sum to 360, the last angle at X must also have measure $90°$.) Then opposite sides $\overline{JK}$ and $\overline{XZ}$ are congruent and $JK = XZ$. Note that DK is given and $JK = DK - DJ = 60 - DJ$. The goal is to find DJ; then we can find JK and hence XZ.

Note that ΔDJX and ΔXYD are both right triangles that share a side. They are congruent, as proven below, and therefore $\overline{DJ} \cong \overline{XY}$ because they are corresponding sides of congruent triangles. Skipping to the answer, $DJ = XY = 45$, so $XZ = JK = 60 - DJ = 60 - 45 = \boxed{15}$.

We need only find one more pair of congruent angles to prove that $\Delta DJX \cong \Delta XYD$ and complete the justification. Proceeding, recall that each diagonal divides a rectangle into two congruent triangles, and the triangles created by one diagonal are congruent to the triangles created by the other diagonal. In particular, $\Delta BDC \cong \Delta ACD$. Then $\angle BDC \cong \angle ACD$ because they are corresponding parts. Note also that $\angle JXD \cong \angle ACD$ because they are corresponding angles formed by transversal $\overline{DC}$ of parallel lines JX and AC. Then by the transitive property $\angle JXD \cong \angle BDC$. Finally, $\Delta DJX \cong \Delta XYD$ by AAS.

Round 4 - Sequences and Series

1. Find all values of n such that the sum of the first n terms of sequence $10, 8, 6, 4, ...$ equals 18.

Solution: Note that the first two terms sum to 18: $10+8=18$. Clearly, one value is $n=2$. Also, note that the sum of terms will increase as long as the next term is positive, and it will decrease if the next term is negative. So, the sum of the sequence will increase after adding the third, fourth, and fifth terms, and then will decrease, starting with the seventh term (-2). Finally, note that the sum of the third, fourth, and fifth, terms $(6+4+2)$ is exactly cancelled by the sum of the ninth, eighth, and seventh terms $(-6-4-2)$. Therefore the sum of the first nine terms equals the sum of the first two terms, and another value is $n=9$. Using set notation: $n \in \{\boxed{2, 9}\}$.

2. A ball is dropped from 10 feet above the ground. Each time it bounces, it rebounds to $\frac{4}{5}$ of the height from which it fell. How far, in feet, will the ball travel before coming to rest?

Solution: The ball travels 10 feet down when it is dropped. It then travels $10\frac{4}{5} = 8$ feet up on the first bounce, and 8 feet back down to the drown. On successive bounces the ball travels $2(10\left(\frac{4}{5}\right)^2)$, $2(10\left(\frac{4}{5}\right)^3)$, and $2(10\left(\frac{4}{5}\right)^n)$ for the n^{th} bounce; twice the height of the bounce. Thus the total distance travelled is 10 feet plus an infinite geometric series starting with 16 feet (twice the height of the first bounce) and common ratio $\frac{4}{5}$. Applying the formula for a geometric series with first element x_1, common ratio r and sum S:

$$\begin{aligned} S &= \frac{x_1}{1-r} \\ &= \frac{16}{1-\frac{4}{5}} \\ &= \frac{16}{\frac{1}{5}} \\ &= 16 \cdot 5 = 80 \end{aligned}$$

and the total distance is $10 + 80 = \boxed{90}$ feet.

3. How many terms of the arithmetic sequence $1, \frac{17}{15}, \frac{19}{15}, ...$ must be summed to equal the sum of the first five terms of the geometric sequence $3, 6, 12, ...$?

Solution: The sum of the first five terms of the given geometric sequence is $3+6+12+24+48 = 93$. The common difference of the arithmetic sequence is $\frac{2}{15}$, and the n^{th} term is $1+(n-1)\frac{2}{15} = \frac{13}{15}+\frac{2n}{15} = \frac{13+2n}{15}$. The sum of the first n terms of the arithmetic sequence is n times the average of the first term (1) and the n^{th} term:

$$S = n\frac{1+\frac{13+2n}{15}}{2} = n\frac{\frac{15+13+2n}{15}}{2} = n\frac{28+2n}{2 \cdot 15} = n\frac{14+n}{15}$$

Set this ratio equal to 93 to create quadratic that can be solved for n:

$$n\frac{14+n}{15} = 93$$

Multiplying both sides by 15 and subracting $15 \cdot 93 = 1395$ from both sides yields:

$$n^2 + 14n - 1395 = 0$$

This quadratic can be factored because $-1395 = 15(-93) = 45(-31)$ and we have found two numbers that sum to 14 and whose product is 1395. The factored form is:

$$(n-31)(n+45) = 0$$

and the only positive solution is $n = \boxed{31}$.

Round 5 - Matrices and Systems of Equations

1. If $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, find AB.

Solution: Note that A is a 2 X 2 (2 rows, 2 columns) matrix and B is a 2 X 3 matrix. The matrix product AB exists because the number of columns in A equals the number of rows in B. The dimensions of AB are 2 X 3 matrix because A has 2 rows and B has 3 columns. By the dot product definition, each element of AB is equal to the dot product of the corresponding row of A with the corresponding column of B:

$$AB = \begin{bmatrix} 5\cdot 1 + 4\cdot 0 & 5\cdot 0 + 4\cdot 1 & 5\cdot 2 + 4\cdot 4 \\ 2\cdot 1 + 3\cdot 0 & 2\cdot 0 + 3\cdot 1 & 2\cdot 2 + 3\cdot 4 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 4 & 26 \\ 2 & 3 & 16 \end{bmatrix}}$$

2. Find the determinant of $A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & -1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution: Use cofactor expansion to expand on the first row of A:

$$\begin{aligned} \begin{vmatrix} 1 & 4 & 1 \\ 2 & 5 & -1 \\ 1 & 1 & 2 \end{vmatrix} &= 1\cdot \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} - 4\cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + 1\cdot \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} \\ &= 1\cdot(5\cdot 2 - (-1)\cdot 1) - 4\cdot(2\cdot 2 - (-1)\cdot 1) + 1\cdot(2\cdot 1 - 5\cdot 1) \\ &= (10+1) - 4(4+1) + (2-5) \\ &= 11 - 20 - 3 = \boxed{-12} \end{aligned}$$

3. Solve the following system of linear equations. Write your solution as the ordered triple (x, y, z).

$$\begin{cases} -2x + y + 2z = 3 \\ 4x + 2y + z = 2 \\ 6x + 8y + 7z = 0 \end{cases}$$

Solution: There are many ways to solve this system using a variety of eliminations and substitutions. We will start by forming the augmented matrix:

$$\begin{bmatrix} -2 & 1 & 2 & 3 \\ 4 & 2 & 1 & 2 \\ 6 & 8 & 7 & 0 \end{bmatrix}$$

As shown above, the augmented matrix consists of the coefficients in the first three columns and the right hand side constants in the last column. Then perform elementary row options, adding or subtracting muliples of one row to another, until it is transformed into reduced row echelon form. This process is known as Gauss-Jordan reduction, or Gauss-Jordan elimination. Start by eliminating the first variable x from the second two equations in two steps: add twice the first row $(2r_1)$ to the second row (r_2), then add $3r_1$ to r_3 (the third row):

$$\begin{bmatrix} -2 & 1 & 2 & 3 \\ 4 & 2 & 1 & 2 \\ 6 & 8 & 7 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 2 & 3 \\ 0 & 4 & 5 & 8 \\ 6 & 8 & 7 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 2 & 3 \\ 0 & 4 & 5 & 8 \\ 0 & 11 & 13 & 9 \end{bmatrix}$$

For the next three steps, subtract $2r_2$ from r_3; subtract r_3 from r_2; and

$$\longrightarrow \begin{bmatrix} -2 & 1 & 2 & 3 \\ 0 & 4 & 5 & 8 \\ 0 & 3 & 3 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 2 & 3 \\ 0 & 1 & 2 & 15 \\ 0 & 3 & 3 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 2 & 3 \\ 0 & 1 & 2 & 15 \\ 0 & 0 & -3 & -52 \end{bmatrix}$$

The next four steps: subtract r_2 from r_1; divide r_3 by -3; subtract $2r_3$ from r_2; and divide r_1 by 2:

$$\longrightarrow \begin{bmatrix} -2 & 0 & 0 & -12 \\ 0 & 1 & 2 & 15 \\ 0 & 0 & -3 & -52 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 0 & 0 & -12 \\ 0 & 1 & 2 & 15 \\ 0 & 0 & 1 & \frac{52}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 0 & 0 & -12 \\ 0 & 1 & 0 & -\frac{59}{3} \\ 0 & 0 & 1 & \frac{52}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -\frac{59}{3} \\ 0 & 0 & 1 & \frac{52}{3} \end{bmatrix}$$

The system has been transformed to the three equations: $x = 6$, $y = -\frac{59}{3}$, and $z = \frac{52}{3}$, so the answer is: $(x, y, z) = \boxed{(6, -\frac{59}{3}, \frac{52}{3})}$. This answer can be checked by plugging these values into the original system.

Team Round

1. Joshua sold a bicycle for \$231, making a 10% profit on what he paid for the bike. At what price, in \$, would he have needed to sell it to make a 30% profit instead?

Solution: Let C be defined as the amount that Joshua paid for the bicycle, which is the cost to Joshua of the bicycle. Joshua's profit is equal to 10% of C since it is calculated as a percentage of Joshua's cost, or $\frac{10}{100}C = 0.1C$. The sales price equals the sum of the cost and the profit for the bicycle, or $\$231 = C + 0.1C = 1.1C$. Thus, $C = \$231 \div 1.1 = \$210$.

If Joshua were to sell the bicycle at a 30% profit instead, he would need to sell it at a price equal to $C + 0.3C = 1.3C$, since $30\% = \frac{30}{100} = 0.3$. This price is $1.3 \cdot \$210 = \$\boxed{273}$.

2. Tim paints fences at a rate of once fence every 5 hours. Jim paints fences at a rate of one fence every 4 hours. When Tim and Jim work together, Jim works at his usual rate, but Tim wastes time and paints fewer fences than he would at his usual rate. If Tim and Jim painted 4 fences in 10 hours working together, how much time did Tim waste? Express your answer in minutes.

Solution: Jim paints $\frac{10}{4} = 2.5$ fences in ten hours working at his normal rate. That leaves $4 - 2.5 = 1.5$ fences that were painted by Tim. Tim would have taken $\frac{1.5}{\frac{1}{5}} = 1.5 \cdot 5 = 7.5$ hours working at his normal rate to paint 1.5 fences. Therefore Tim wasted $10 - 7.5 = 2.5$ hours. Converting hours to minutes, Tim wasted $2.5 * 60 = \boxed{150}$ minutes.

3. The measure in degrees of an interior angle of a regular polygon is $90°$ less than 8 times the measure of an exterior angle. Each of the polygon's legs is 6 cm long. Find the perimeter of the polygon, in cm.

Solution: Let the number of sides of the polygon be n so that the perimeter of the polygon is $6n$ cm. To find the perimeter, find n.

The measure of an exterior angle (x) of an n sided polygon is $x = \frac{360}{n}°$ because the sum of the exterior angles amounts to a complete revolution around the polygon. The interior and exterior angles are supplementary because they form a line. Therefore the measure of an interior angle (t) is equal to $t = 180 - \frac{360}{n}°$. (Note, this is equal to the expression $\frac{180(n-2)}{n}$). The problem states that $t = 8x - 90°$. Substitute the two expressions into this equation and solve for n:

$$\begin{aligned} 180 - \frac{360}{n} &= 8\frac{360}{n} - 90 \\ 180 + 90 &= (1+8)\frac{360}{n} \\ 270 &= 9\frac{360}{n} \\ n &= 9\frac{360}{270} = 9\frac{4}{3} = 3 \cdot 4 = 12 \end{aligned}$$

The perimeter equals $6n$ cm $= 6(12) = \boxed{72}$ cm.

4. Find the product defined below and express your answer in simplest form.

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdot \ldots \cdot \left(1+\frac{1}{100}\right)$$

Solution: Let x be the desired product. Write the first few terms, observe the pattern, and note that the product can be "telescoped":

$$\begin{aligned} x =& \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdot \ldots \cdot \left(1+\frac{1}{99}\right)\left(1+\frac{1}{100}\right) \\ =& \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot \ldots \cdot\frac{100}{99}\cdot\frac{101}{100} \end{aligned}$$

The numerator in the first term cancels the denominator in the second term, and the numerator in the second term cancels the denominator in the third term. The n^{th} term in the product is $\left(1+\frac{1}{n}\right) = \left(\frac{n+1}{n}\right)$; its numerator will cancel the denominator of the (preceding) $(n-1)^{th}$ term, and its denominator with cancel the numerator of the (succeeding) $(n+1)^{th}$ term. In fact, the numerator and denominator of every middle term will be cancelled, leaving only the denominator of the first term and the numerator of the 100^{th} term. Thus $x = \frac{101}{1} = \boxed{101}$.

5. Let $A = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 4 & 5 \end{bmatrix}$. Find the matrix product $A^{-1}BA$, expressed as a single matrix.

Solution: The inverse of a 2 X 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then $A^{-1} = \dfrac{1}{3(-1)}\begin{bmatrix} -1 & 0 \\ -2 & 3 \end{bmatrix}$. Because matrix multiplication satisfies the associative property, $A^{-1}BA = (A^{-1}B)A = A^{-1}(BA)$. Computing starting with the last product:

$$BA = \begin{bmatrix} -1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1(3)+0(2) & -1(0)+0(-1) \\ 4(3)+5(2) & 4(0)+5(-1) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 22 & -5 \end{bmatrix}$$

$$A^{-1}(BA) = -\frac{1}{3}\begin{bmatrix} -1 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 22 & -5 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1(-3)+0(22) & -1(0)+0(-5) \\ -2(-3)+3(22) & -2(0)+3(-5) \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 3 & 0 \\ 3(24) & 3(-5) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -1 & 0 \\ -24 & 5 \end{bmatrix}}$$

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6. A rabbit family consists of males and females. Each male in the family has one fewer female relatives than he has male relatives. Each female has two fewer male relatives than twice the number of her female relatives. What is the total number of rabbits (males and females) in the family?

Solution: Let m equal the number of males and f equal the number of females in the family. Note that a bunny is not a relative of itself, so that each male has $m-1$ male relatives and f female relatives, while each female has m relatives and $f-1$ female relatives. Then the given information can be translated directly into the following two equations:

$$f = (m-1) - 1$$
$$m = 2(f-1) - 2$$

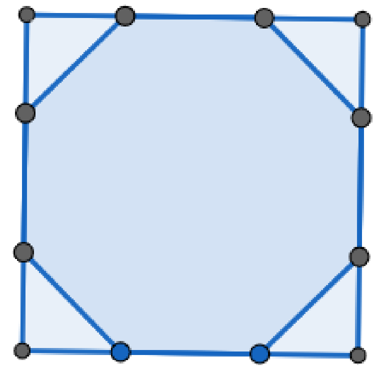
Now substitute $f = m-2$ (the first equation) into the second equation and solve for m:

$$m = 2(m-2-1) - 2 = 2m - 6 - 2 = 2m - 8$$

Add 8 and subtract m from each side of $m = 2m-8$, leaving $m = 8$. Then $f = m - 2 = 8 - 2 = 6$, and the total number of rabbits is $m + f = 8 + 6 = \boxed{14}$.

7. A miniature stop sign in the shape of a regular octagon can be cut from a 4in X 4in square sheet of metal. The maximum area for the stop sign can be expressed as $a + b\sqrt{3}$ in^2. Find the ordered triple (a, b, c).

Solution: Note that the external angles in a regular octagon measure $\frac{360}{8} = 45°$. Therefore each opposite pair of sides are parallel (four external angles together measure $4 \cdot 45 = 180°$) and alternate pairs of opposite sides are perpendicular (two external angles together measure $2 \cdot 45 = 90°$. Furthermore, the distance between opposite sides is the same for each of the four pairs. As a result, a regular octagon fits exactly inside a square when two pairs of opposite sides lie on two opposite sides of the square, as shown in the figure at the right.



The four triangles cut from the square are right isosceles triangle because the corner angle measures $90°$ and the other two angles are external angles of the octagon and they measure $45°$. The area of the square is $4^2 = 16$in^2, and the area of the octagon equals the area of the square minus the areas of the four triangles. Let s be the side length of the stop sign (octagon). Then the length of the hypotenuse of the cutout triangles is s and the leg lengths are $\frac{\sqrt{2}}{2}$. If T is the area of one triangle, then $T = \frac{1}{2}\left(s\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}s^2\frac{1}{2} = \frac{s^2}{4}$. The total cutout area is $4T = 4\frac{1}{4}s^2 = s^2$. Thus, the area of the octagonal stop sign is $(16 - s^2)$in^2. It remains to find s.

Note that the side length of the square is 4in, and is equal to two legs of the cutout triangle plus s, or $4 = 2s\frac{\sqrt{2}}{2} + s = s(\sqrt{(2)} + 1)$. Therefore:

$$\begin{aligned} s &= \frac{4}{\sqrt{2}+1} \\ &= \frac{4(\sqrt{2}-1)}{(\sqrt{2}+1)l(\sqrt{2}-1)} \\ &= \frac{4\sqrt{2}-4}{(\sqrt{2})^2 - 1^2} \\ &= \frac{4\sqrt{2}-4}{(2-1)} \\ &= 4\sqrt{2} - 4 \end{aligned}$$

Then $s^2 = (4\sqrt{2}-4)^2 = (4\sqrt{2})^2 + 2(4)(-4)\sqrt{2} + (-4)^2 = 16 \cdot 2 - 32\sqrt{2} + 16 = 32 + 16 - 32\sqrt{2}$
Finally, the area of the octagonal stop sign is $16 - 32 - 16 + 32\sqrt{2} = -32 + 32\sqrt{2}$ and $(a, b, c) = \boxed{(-32, 32, 2)}$.

8. A contest winner was offered the choice of two prizes. The first prize consisted of 12 daily cash payments: \$160 on the first day, \$190 on the second day, \$220 on day 3, with the daily cash payment increasing by \$30 each day up to day 12. The other prize consisted of 12 daily cash payments: \$1 on day 1, \$2 on day 2, \$4 on day 3, doubling each day up to and including day 12. What is the absolute value of the difference (in \$) between the two prizes?

Solution: The first prize is the sum of the first 12 terms of the arithmetic sequence 160, 190, 220, ..., 490 ($490 = 160 + 11 \cdot 30$ is the payment on day 12). This sum is 12 times the average of the first and last terms in the sequence, or $12\frac{160+490}{2} = 6 \cdot 650 = 3900$. Thus, the total amount of the first prize is \$3900. The second prize is the sum of the geometric sequence $1, 2, 4, ..., 2^{12-1}$, in dollars. Using the formula for a geometric series with ratio $r = 2$ and number of terms $n = 12$, thetotal amount of the second prize S is:

$$S = \frac{1 - 2^{12}}{1 - 2} = \frac{1 - 4096}{-1} = 4096 - 1 = \$4095$$

Therefore the absolute value of the difference in the two prizes is $|3900 - 4095| = \$\boxed{295}$.

9. At the Pittsburgh zoo, children ride a train for \$0.50 and adults ride the train for \$2. On a given day, 1088 passengers paid a total of \$1090. How many of those passengers were children?

Solution: Let c be the number of child passengers and a be the number adult passengers. Then the given information is captured in the following two equations:

$$\begin{aligned} a + c &= 1088 \\ 2a + 0.5c &= 1090 \end{aligned}$$

Now eliminate a, the number of adult passengers, by subtracting the second equation from two times the first equation:

$$\begin{array}{rr} & 2\text{a} + \quad 2\text{c} = 2176 \\ - & (2\text{a} + 0.5\text{c} = 1090) \\ \hline & 1.5c = 1086 \end{array}$$

Next, multiply the equation by 2: $3c = 2172$. Solving $c = \frac{2172}{3} = \boxed{724}$.